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M.Sc DEGREE (CSS) EXAMINATION, MARCH 2026

Fourth Semester

M Sc MATHEMATICS

ELECTIVE - ME800401 - DIFFERENTIAL GEOMETRY

2019 Admission Onwards

15579151

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Find the level set of the function $f(x_1, x_2) = x_1^2 + x_2^2$ for $c = -2, 0, 2, 5$
2. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is contained in the n-dimesional vector subspace $[\nabla f(p)]^{\perp}$ of $\mathbb{R}_p^{n+1}$ consisting of all vectors orthogonal to $\nabla f(p)$ if $\nabla f(p) \neq 0$.
3. Describe the spherical image, when $n = 1$, of $x_1^2 - x_2^2 - \ldots - x_{n+1}^2 = 4, x_1 > 0$ oriented by $\mathbf{N} = \frac{\nabla f}{\|\nabla f\|}$.
4. Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (t, t^2)$.
5. Define Euclidean and Levi - Civita parallelism.
6. Define covariant derivative of the tangent vector field $\mathbf{X}$ with respect to $\mathbf{v} \in S_p$. Show that $D_{\mathbf{v}}(\mathbf{X} + \mathbf{Y}) = D_{\mathbf{v}}\mathbf{X} + D_{\mathbf{v}}\mathbf{Y}$, for all smooth tangent vector fields $\mathbf{X}$ and $\mathbf{Y}$ on S.
7. Write a formula for finding curvature of a plane curve at the point p. Also define curvature of a plane curve.
8. Let U be an open set in $\mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}$ is smooth. Then for any parametrized curve $\alpha : I \to U$ evaluate $\int_{\alpha} df$.
9. Define parametrized n-surface in $\mathbb{R}^{n+k}(k \geq 0)$. Give an example.

10. Define vector field and coordinate vector fields along a smooth map $\varphi : U \to \mathbb{R}^{n+k}$, where U open in $\mathbb{R}^n$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. State and prove Maximal Integral Curve theorem for open subsets in $\mathbb{R}^{n+1}$.

12. For what values of c, the level set $f^{-1}(c)$ is an n-surface, where
    a) $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + \ldots + x_{n+1}^2$
    b) $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2$

13. Verify the following three properties of the differentiation of vector fields along parametrized curves:
    a) $\mathbf{X} \dot{+} \mathbf{Y} = \dot{\mathbf{X}} + \dot{\mathbf{Y}}$
    b) $f\dot{\mathbf{X}} = f'\mathbf{X} + f\dot{\mathbf{X}}$
    c) $(\mathbf{X} \cdot \mathbf{Y})' = \dot{\mathbf{X}} \cdot \mathbf{Y} + \mathbf{X} \cdot \dot{\mathbf{Y}}$

14. Let $\alpha : [0, \pi] \to S^2$ be the half great circle in S^2, running from the north pole $p = (0,\ 0,\ 1)$ to the south pole $q = (0,\ 0, -1)$, defined by $\alpha(t) = (sint,\ 0,\ cost)$. Show that, for $\mathbf{v} = (p,\ v_1,\ v_2,\ 0) \in S_p^2,\ P_\alpha(\mathbf{v}) = (q,\ -v_1,\ v_2,\ 0)$

15. Let $\alpha(t) = (x(t), y(t)), t \in I$ be a local parametrization of the oriented plane curve C. Show that $\kappa \circ \alpha = \frac{(x'y'' - y'x'')}{(x'^2 + y'^2)^{3/2}}$.

16. Define length of a connected oriented plane curve. Find the length of the connected oriented plane curve $f^{-1}(c)$ oriented by $\frac{\nabla f}{\|\nabla f\|}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2, c = r^2$

17. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Let $\mathbf{Z}$ be any non-zero normal vector field on S such that $\mathbf{N} = \mathbf{Z}/\|\mathbf{Z}\|$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for S_p. Prove that

$$\mathbf{K}(p) = \frac{(-1)^n det \begin{pmatrix} \nabla_{\mathbf{v}_1}\mathbf{Z} \\ \vdots \\ \nabla_{\mathbf{v}_n}\mathbf{Z} \\ \mathbf{Z}(p) \end{pmatrix}}{\|\mathbf{Z}(p)\|^n det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{Z}(p) \end{pmatrix}}$$

18. *Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$. Prove that the Gauss – Kronecker curvature $\mathbf{K}(p)$ of S at p is non-zero for all $p \in S$ if and only if the second fundamental form $\mathcal{S}_p$ of S at p is definite for all $p \in S$.*

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Prove the following if S is an n-surface in $\mathbb{R}^{n+1}$ and $\mathbf{X}$ is a smooth tangent vector field on S with $p \in S$. There exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that(i) $\alpha(0) = p$ $(ii)\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ for all $t \in I$ and (iii) If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$

20. Given S is a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\bigtriangledown f(p) \neq 0, \forall p \in S$. Is the Gauss map from S to the unit sphere S^n onto ? Explain.

21. Prove that the Weingarten map L_p is self-adjoint; that is , $L_p(\mathbf{v}).\mathbf{w} = \mathbf{v}.L_p(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in S_p$.

22. *Prove the following.*

a) Let V be a finite dimensional vector space with dot product and let $L: V \to V$ be a self adjoint linear transformation on V. Let $S = \{v \in V: v \cdot v = 1\}$ and define $f: S \to \mathbb{R}$ by $f(v) = L(v) \cdot v$. If f is stationary at $v_0 \in S$, then $L(v_0) = f(v_0) \cdot v_0$.

b) Let V be a finite dimensional vector space with dot product and let $L: V \to V$ be a self adjoint linear transformation on V. Then there exists an orthonormal basis for V consisting of eigen vectors of L.

(2×5=10 weightage)